

*Chapter 1***ASTEROIDS DIMENSIONS AND THE TRUNCATED
PARETO DISTRIBUTION***Lorenzo Zaninetti* *

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PACS 96.30.Ys **Keywords:** Asteroids**Abstract**

In this chapter first the statistics of the standard and truncated Pareto distributions are derived and used to fit empirical values of asteroids diameters from different families, namely, Koronis, Eos and Themis, and from the Astorb database. A theoretical analysis is then carried out and two possible physical mechanisms are suggested that account for Pareto tails in distributions of asteroids diameter.

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1. Introduction

The interest in the study of asteroids in the inner solar system lays in their connection with the formation of planets and their temporal evolution.

Among others, studies on asteroids formations and evolutions involve

1. the effects of asteroid collisional history on sizes and spins of present-day objects [1],
2. realistic collisional scaling laws and the implications of including observables, such as collisional produced families, in constraining the collisional history of main-belt asteroids [2],
3. the creation of a model of the main asteroid belt whose purpose is to describe distribution of size size frequency of asteroids and simulate their number [3].
4. temporal evolution for 4.2 Myr of test particles, which were initially placed on a perfectly rectangular grid and subjected to gravitational interactions with the Sun and five planets, from Mars to Neptune , see [4].

On the other hand, it has been shown that experimental observations can be fitted with a differential size distribution

$$n = dN/dD = n(D) \propto D^{-\alpha} \quad , \quad (1)$$

where D is the diameter in Km , α the exponent of the inverse power law and n the number of asteroids comprised between D and $D + dD$. Measurements of the properties of 13,000 asteroids detected by the Sloan Digital Sky Survey (SDSS) present a differential size distribution that for $D \geq 5Km$ is $n \propto D^{-4}$ and for $D \leq 5Km$ is $n \propto D^{-2.3}$, see [4].

The ongoing simulations as well the observations require a careful analysis of the Pareto distribution [5, 6] and the truncated Pareto distribution [7, 8, 9]. This paper presents in Section 2. a comparison between the Pareto and the truncated Pareto distributions. In Section 3. the theoretical results are applied to the distribution of the radius of asteroids. Two physical mechanisms that produces a Pareto type distribution for diameters are presented in Section 4.

2. Statistical Distribution

Let X be a random variable taking values x in the interval $[a, \infty]$, $a > 0$. The probability density function (in the following pdf) named Pareto is defined by [6]

$$f(x; a, c) = ca^c x^{-(c+1)} \quad , \quad (2)$$

$c > 0$, and the Pareto distribution functions is $F(x : a, c) = 1 - a^c x^{-c}$

An upper truncated Pareto random variable is defined in the interval $[a, b]$ and the corresponding pdf is

$$f_T(x; a, b, c) = \frac{ca^c x^{-(c+1)}}{1 - \left(\frac{a}{b}\right)^c} \quad , \quad (3)$$

[9] and the truncated Pareto distribution function is

$$F_T(x; a, b, c) = \frac{1 - \left(\frac{a}{x}\right)^c}{1 - \left(\frac{a}{b}\right)^c} . \quad (4)$$

Moments of the truncated distributions exist for all $c > 0$. For instance, the mean of $f_T(x; a, b, c)$ is, for $c \neq 1$ and $c = 1$, respectively,

$$\langle x \rangle = \frac{ca}{c-1} \frac{1 - \left(\frac{a}{b}\right)^{c-1}}{1 - \left(\frac{a}{b}\right)^c}, \quad \langle x \rangle = \frac{ca^c}{1 - \left(\frac{a}{b}\right)^c} \ln \frac{b}{a} \quad (5)$$

Similarly, if $c \neq 2$, the variance is given by

$$\sigma^2 = \frac{ca^2}{(c-2)} \frac{1 - \left(\frac{a}{b}\right)^{c-2}}{1 - \left(\frac{a}{b}\right)^c} - \langle x \rangle^2, \quad (6)$$

whereas for $c = 2$

$$\frac{ca^c}{1 - \left(\frac{a}{b}\right)^c} \ln \frac{b}{a} - \langle x \rangle^2 . \quad (7)$$

In general the $n - th$ central moment is

$$\begin{aligned} & \int_a^b (x - \langle x \rangle)^n f_T(x) dx = \\ & (-\langle x \rangle)^n a^{-c} {}_2F_1(-c, -n; 1 - c; \frac{a}{\langle x \rangle}) \left((a^c)^{-1} - (b^c)^{-1} \right)^{-1} \\ & - (-\langle x \rangle)^n b^{-c} {}_2F_1(-c, -n; 1 - c; \frac{b}{\langle x \rangle}) \left((a^c)^{-1} - (b^c)^{-1} \right)^{-1} \end{aligned} \quad (8)$$

where ${}_2F_1(a, b; c; z)$ is a regularized hypergeometric function, see [10, 11, 12]. An analogous formula based on some of the properties of the incomplete beta function, see [13] and [14], can be found in [15]. The median m of the Pareto distribution is

$$m = 2^{c-1} a , \quad (9)$$

and the median of truncated Pareto m_T

$$m_T = a 2^{c-1} (a^c b^{-c} + 1)^{-c-1} . \quad (10)$$

Parameters of the truncated Pareto pdf from empirical data can be obtained via the maximum likelihood method; explicit formulas for maximum likelihood estimators (MLE) are given in [7], and for the more general case in [9], whose results we report here for completeness.

Consider a random sample $\mathcal{X} = x_1, x_2, \dots, x_n$ and let $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(n)}$ denote their order statistics so that $x_{(1)} = \max(x_1, x_2, \dots, x_n)$, $x_{(n)} = \min(x_1, x_2, \dots, x_n)$.

The MLE of the parameters a and b are

$$\tilde{a} = x_{(n)}, \quad \tilde{b} = x_{(1)}, \quad (11)$$

respectively, and \tilde{c} is the solution of the equation

$$\frac{n}{\tilde{c}} + \frac{n \left(\frac{x_{(n)}}{x_{(1)}} \right)^{\tilde{c}} \ln \left(\frac{x_{(n)}}{x_{(1)}} \right)}{1 - \left(\frac{x_{(n)}}{x_{(1)}} \right)^{\tilde{c}}} - \sum_{i=1}^n [\ln x_i - \ln x_{(n)}] = 0, \quad (12)$$

[9].

There exists a simple test to see whether a Pareto model is appropriate [9]: the null hypothesis $H_0 : \nu = \infty$ is rejected if and only if $x_{(1)} < [nC/(-\ln q)]^{1/c}$, $0 < q < 1$, where $C = a^c$. The approximate p -value of this test is given by $p = \exp \left\{ -nC x_{(1)}^{-c} \right\}$, and a small value of p indicates that the Pareto model is not a good fit.

Given a set of data is often difficult to decide if they agree more closely with f or f_T , since, in the interval $[a, b]$, they differ only or a multiplicative factor $1 - (a/b)^c$, that approaches 1 even for relatively small values of c if the interval $[a, b]$ is not too small. For this reason, rather than f and f_T , the distributions $P(X > x)$ and $P_T(X > x)$ are used, often called survival functions, that are given respectively by

$$P(X > x) = S(x) = 1 - F(x; a, c) = a^c x^{-c} \quad (13)$$

and

$$P_T(X > x) = S_T(x) = 1 - F_T(x; a, b, c) = \frac{ca^c (x^{-c} - b^{-c})}{1 - \left(\frac{a}{b} \right)^c}. \quad (14)$$

The Pareto variate X can be generated by

$$X : a, c \sim a (1 - R)^{-\frac{1}{c}}, \quad (15)$$

and the truncated Pareto variate X_T by

$$X_T : a, b, c \sim a \left(1 - R \left(1 - \left(\frac{a}{b} \right)^c \right) \right)^{-\frac{1}{c}}, \quad (16)$$

where R is the unit rectangular variate.

3. Application to the asteroids

We have tested the hypothesis that diameters of asteroids follows a Pareto distribution by considering different families of asteroids, namely, Koronis, Eos and Themis.

The sample parameter of the families are reported in Table 1, Table 2 and Table 3, whereas Figure 1, Figure 2, Figure 3 report the graphical display of data and the fitting distributions.

Table 1. *Coefficients of diameter distribution of the Koronis family. The parameter c is derived through MLE and $p=0.033$.*

a [km]	b [km]	c	n	$P(X > x)$
25.1	44.3	3.77	29	truncated Pareto
25.1	∞	5.04	29	Pareto

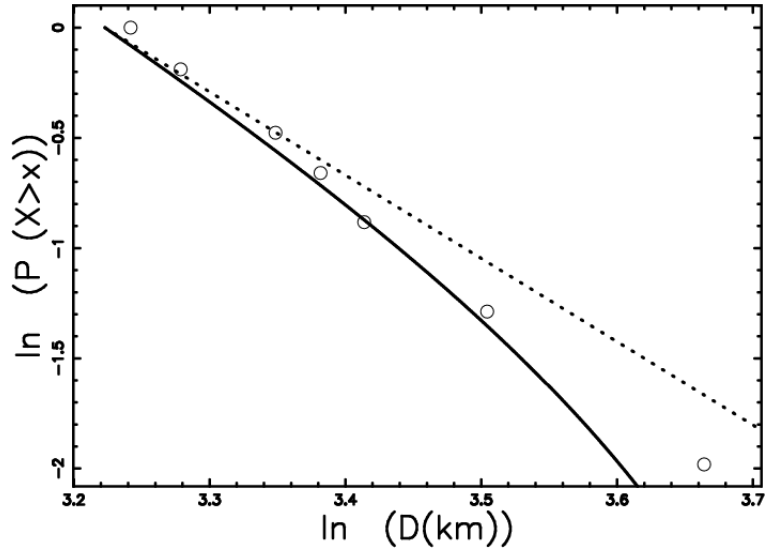


Figure 1. In-In plot of the survival function of diameter distribution of the Koronis Family: data (empty circles), survival function of the truncated Pareto pdf (full line) and survival function of the Pareto pdf (dotted line). A complete sample is considered with parameters as in Table 1.

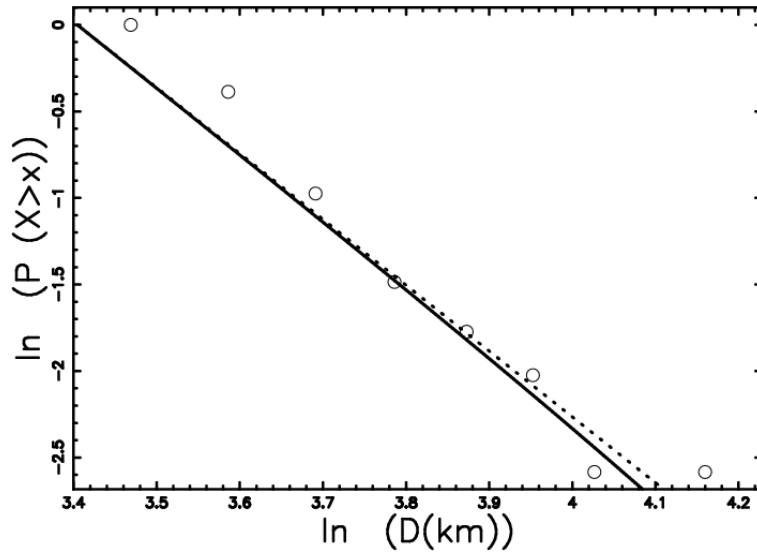


Figure 2. In-In plot of the survival function of diameter distribution of the Eos Family: data (empty circles), survival function of the truncated Pareto pdf (full line) and survival function of the Pareto pdf (dotted line). A complete sample is considered with parameters as in Table 2.

Table 2. *Coefficients of diameter distribution of the Eos family . The parameter c is derived through MLE and $p=0.681$.*

a [km]	b [km]	c	n	$P(X > x)$
30.1	110	3.80	53	truncated Pareto
30.1	∞	3.94	53	Pareto

Table 3. *Coefficients of diameter distribution of the Themis family . The parameter c is derived through MLE and $p=0.67$.*

a [km]	b [km]	c	n	$P(X > x)$
35.3	249	2.5	53	truncated Pareto
35.3	∞	2.6	53	Pareto

In case of the Koronis family P_T fits the data better than P and indeed $p = 0.039$ is correspondingly small, whereas for the Eos family, P performs slightly better than P_T ($p=0.68$), and the estimated of c are very closed in both cases. Finally in the third case, the Themis family, the two distributions are the same, due to the fact that the ratio $a/b = 0.14$ is small.

Another interesting catalog is the Asteroid Orbital Elements Database (Asterdb) which is visible at <http://vizier.u-strasbg.fr/>; the sample parameter of the asteroids with diameter > 90 Km is reported in Table 4 and the fitting survival function in Figure 4.

Table 4. *Coefficients of diameter distribution of the asteroids extracted from Asterdb database with diameter > 90 Km. The parameter c is derived through MLE and $p=0.53$.*

a [km]	b [km]	c	n	$P(X > x)$
90.59	848.4	2.71	272	truncated Pareto
90.59	∞	2.75	272	Pareto

4. Simulating Pareto tails

Two models that explains the Pareto tails are presented. The first analyzes the possibility that the asteroids are formed from smaller bodies and the second one introduces a fragmentation model at the light of the Voronoi diagrams.

4.1. Accretion

As a simple example of how a distribution with power can be generated, consider the growth of a primeval nebula via accretion, that is the process by which nebulae “capture” mass. We start by considering an uniform pdf for the initial mass of N primeval nebulae, m , in a range $m_{min} < m \leq m_{max}$. At each interaction the i -th nebula has a probability λ_i to increase its mass m_i that is given by

$$\lambda_i = (1 - \exp(-akm_i)), \quad (17)$$

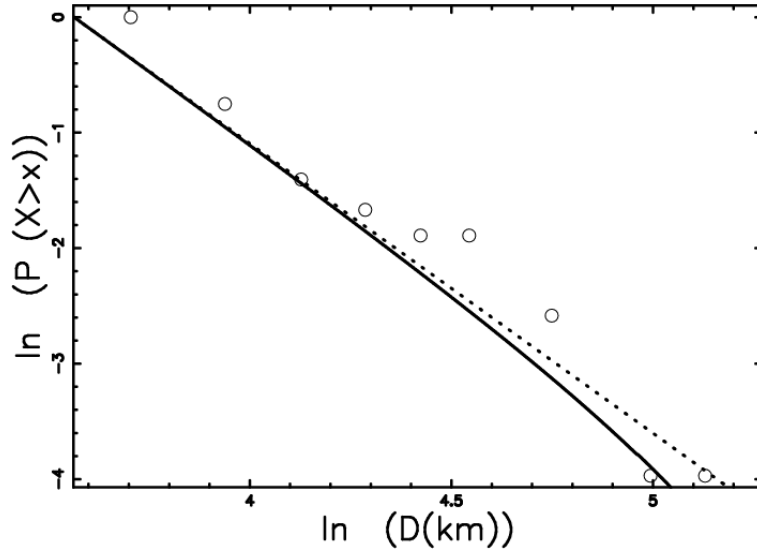


Figure 3. In-In plot of the survival function of diameter distribution of the Themis Family: data (empty circles), survival function of the truncated Pareto pdf (full line) and survival function of the Pareto pdf (dotted line). A complete sample is considered with parameters as in Table 3.

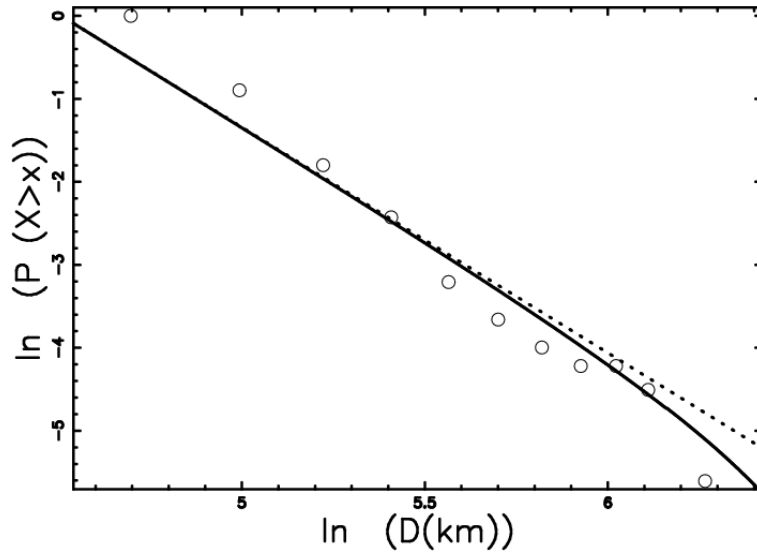


Figure 4. In-In plot of the survival function of diameter distribution of the asteroids extracted from Astorb database with diameter $> 90 \text{ Km}$, data (empty circles), survival function of the truncated Pareto pdf (full line) and survival function of the Pareto pdf (dotted line). A complete sample is considered with parameters as in Table 4.

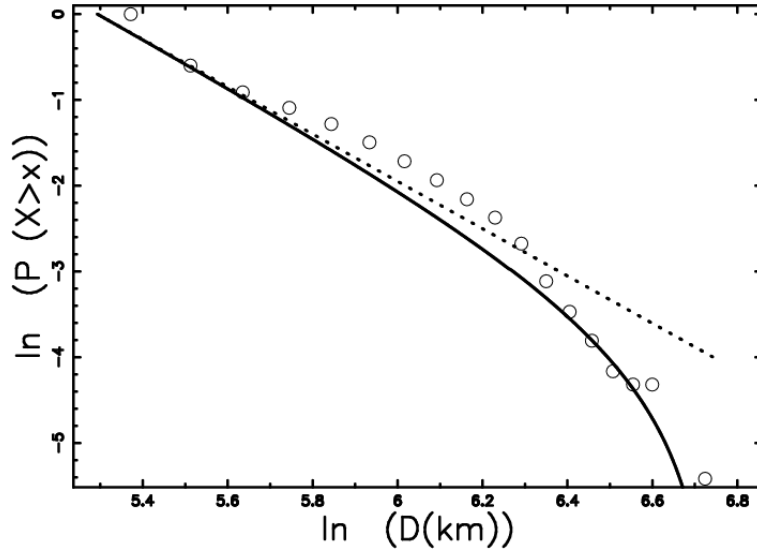


Figure 5. log-log plot of the survival function of the diameter distribution for the primeval nebula. The truncated Pareto parameters are $c=2.75$ and $p=0.00026$.

where ak is a parameter of the simulation; thus more “massive” nebulae are more likely to grow, via accretion. The quantity of which the primeval nebula can grow varies with time, to take into account that the total mass available is limited,

$$\delta m(t) = \delta m(0) \exp(-t/\tau) \quad , \quad (18)$$

where $\delta m(0)$ represents the maximum mass of exchange and τ the scaling time of the phenomena. The simulation proceeds as follows: a number r , is randomly chosen in the interval $[0, 1]$ for each nebula, and, if $r < \lambda_i$, the mass m_i is increased by $\delta m(t)$, where t denotes the iteration of the process. The process proceed in parallel : at each temporal iteration all the primeval nebulae are considered.

Results of the simulations have been fitted with both Pareto survival distributions. see Figure 5

4.2. Fragmentation

The distribution of fragments size arising from breaking of material is still subject of research as it depends on the actual fragmentation process. The first model was developed in [16]: it assumes a time dependent probability of fracture of a ring after a critical strain has been reached in the material. The resulting distribution show that the frequency of occurrence of fragments masses follows a cumulative distribution of the form :

$$\frac{N_m}{N} = e^{-\sqrt{\frac{m}{\mu}}} \quad , \quad (19)$$

where N_m is the number of fragments each of whose mass is greater or equal to m , N

is the total number of fragments ,

$$\mu = \frac{\overline{m}}{2} , \quad (20)$$

and \overline{m} is the averaged mass of a fragment.

4.2.1. Fractal distribution of fragments size

In order to generate cells resulting in a fractal distribution of their volumes the following method can be adopted, which generalizes the procedure presented in [17].

Consider a domain \mathcal{D} subdivided into N_0 cubic cells with a linear dimension l that, that in the following, will be referred as zero-order cells. A zero-order cell is divided into k^3 smaller cubes called zero-order elements, with linear dimension l/k and volumes

$$V_1 = \frac{V_0}{k^3}, \quad (21)$$

where V_0 is the volume of zero-order cell. If P is the probability of a zero-order cell to be fragmented, the number N_1 of zero-order elements generated by fragmentation is given by

$$N_1 = P \cdot k^3 N_0, \quad (22)$$

and the number N_{0a} of zero-order cells that have been not fragmented is

$$N_{0a} = (1 - P)N_0. \quad (23)$$

Each zero-order element now becomes a first-order cell that can be fragmented into first-order elements of volumes

$$V_2 = \frac{V_0}{(k^3)^2} \quad (24)$$

and the number of fragmented first-order elements is

$$N_2 = Pk^3 N_1 = (k^3 P)^2 N_0. \quad (25)$$

The number N_{1a} of first order cells that have not been fragmented is given by

$$N_{1a} = k^3 P(1 - P)N_0. \quad (26)$$

Now first-order elements can be considered second-order cells and the procedure repeats itself. The volume of the n th-order cell V_n is

$$V_n = \frac{V_0}{k^{3n}} , \quad (27)$$

and, after fragmentation, the number of n th-order cells N_{na} is

$$N_n = (Pk^3)^n N_{0a} = (Pk^3)^n (1 - P)N_0 \quad (28)$$

Taking the natural logarithm Eqs. (27) and (28) leads to

$$\ln \frac{V_n}{V_0} = -n \ln(k^3), \quad (29)$$

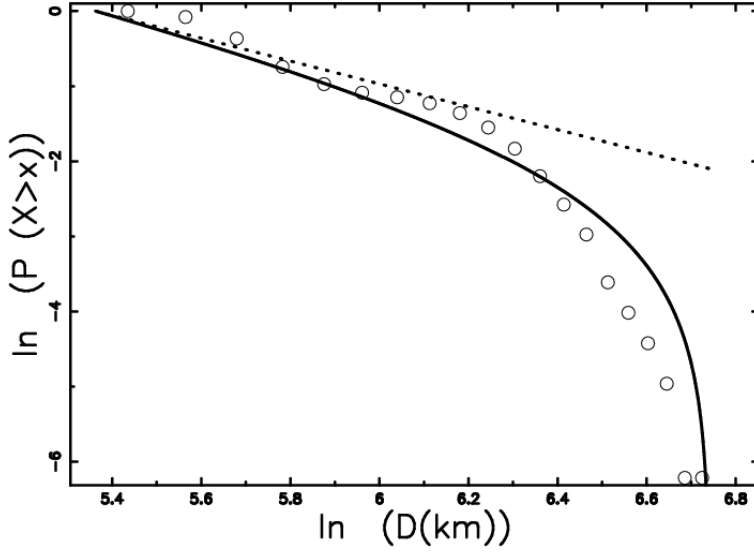


Figure 6. log-log plot of the survival function of the diameter distribution as given by the Voronoi Diagrams in presence of 1000 fractal seeds. The parameters of the simulation are $k=2$, $P=0.92$ and therefore $D_{fr}=2.80$. The truncated Pareto parameters are $c=1.52$ and $p=0$.

$$\ln \frac{N_{na}}{N_{0a}} = -n \ln(Pk)^3 \quad . \quad (30)$$

From Eqs. (30) and (30) it is straightforward to obtain, by elimination of n :

$$\frac{N_{na}}{N_{0a}} = \left[\frac{V_n}{V_0} \right]^{-\frac{\ln[Pk^3]}{\ln[k^3]}} \quad , \quad (31)$$

that is a fractal distribution with dimension D given by

$$D = \frac{3 \ln(Pk^3)}{\ln(k^3)} \quad . \quad (32)$$

Now we can consider the center of cells, of any order, as seeds for the generation of a Voronoi diagram as shown in see Figure 6.

5. Conclusions

In this chapter statistical parameters for the truncated Pareto distribution, namely average, variance, median and n -th central moment have been calculated. Furthermore, also distribution function and survival function have been derived. These quantities allow to fit the various families of asteroids and the Astorb database which are characterized by a finite rather than infinite maximum diameter. Two possible simulations are suggested to produce Pareto tails. The first one results from a simple growth process, in which the increase of the state variable (here mass) depends on the values taken in the previous state. The second

one is a fragmentation process given by 3D Voronoi volumes with a fractal distribution of seeds.

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